Journal of Statistical Physics, Vol. 25, No. 3, 1981

Spontaneous Magnetization of Randomly Dilute Ferromagnets

Hans-Otto Georgii[1]

Received June 11, 1980

We consider Ising ferromagnets on random subgraphs of the square lattice. These are obtained by independent random selections either of sites or of bonds. We assume that for each site (or, respectively, bond) the probability of being selected exceeds the critical percolation probability. Then, at sufficiently low temperatures and zero external field, spontaneous magnetization occurs. Some further related results are obtained.

KEY WORDS: Dilute ferromagnet; random spin system; Ising model; spontaneous magnetization; percolation.

1. INTRODUCTION

In the last two decades, binary alloys which consist of a ferromagnetic (or antiferromagnetic) and a nonmagnetic metal have received an increasing amount of attention. One source of interest are the surprising magnetic properties of the so-called spin glasses[1,2]; their magnetic concentrations are too small to admit a ferromagnetic (or antiferromagnetic) behavior. On the other hand, if the fraction of magnetic atoms is large then it is natural to ask for the effects of the nonmagnetic impurities on the (anti-) ferromagnetic phase transition. In particular, one asks for the critical concentration at which the (anti-) ferromagnetic behavior disappears. Theoretical models of such an alloy are usually of the following type: In a first stage, magnetic and nonmagnetic atoms are distributed at random on the sites of a lattice; then all atoms are frozen in position and, in a second stage, the orientations of the magnetic spins come to equilibrium. Such a two-fold random mechanism is supposed to be a reasonable model of a quenched magnetic

[1] Fakultät für Mathematik, Universität Bielefeld, Bielefeld, West Germany. Present address: Mathematisches Institut der Universität München, Munich, West Germany.

crystal in which the motion of the impurities is slow compared with the magnetic relaxation. A particularly simple model is the dilute Ising ferromagnetic; here the distribution of spins and impurities is governed by a Bernoulli law, and the spins (which are either up or down) interact via an attractive nearest-neighbor potential. This model was extensively studied, for instance, in Refs. 3–10 and the references therein. However, many results are restricted to Bethe lattices or are not satisfactory from a rigorous point of view. An interesting exception is the following theorem of Griffiths and Lebowitz[6]: Suppose the underlying lattice is the square lattice and, for each site, the probability p of being occupied by a spin exceeds approximately 0.985; then spontaneous magnetization occurs at a finite temperature. However, it was expected that this would happen whenever p is larger than the critical percolation probability (which is approximately 0.590); for heuristic considerations supporting this conjecture see Elliott *et al.*,[4] Frisch and Hammersley,[5] and Essam.[7] Moreover, it was believed[7] that at zero temperature the magnetization is given by the probability that the origin belongs to an infinite cluster. In this paper we give a rigorous proof of these conjectures.

Here is an outline of the paper: From a mathematical point of view, the description of dilute spin systems gives rise to the notion of infinite volume Gibbs measures on random graphs. These random graphs are obtained from a d-dimensional integer lattice L either by removing random sites or by dropping random bonds. (The case of random sites corresponds to the model described above; models with random bonds were already considered in Ref. 8, for instance.) These concepts are introduced in Section 2. Then we ask for conditions on the random graph and the interaction which imply almost sure uniqueness or nonuniqueness of the Gibbs measure. In Section 3 we present our results for the case of random sites. These are as follows. Theorem 3.1: The structure of the set of all Gibbs measures on a subset S of L depends only on the macroscopic shape of S. Theorem 3.2: In the presence of an external field, the Gibbs measure is almost surely unique provided the interaction is ferromagnetic and the distribution of the spin positions is invariant under translations. Theorem 3.3: If the random graph is obtained by an independent thinning of the square lattice, such that for each site the probability of being not removed exceeds the critical site percolation probability, then almost surely there are at least two distinct Gibbs measures with respect to the ferromagnetic Ising potential at zero external field and sufficiently low temperatures. Long-range interactions and more general thinnings are considered in two corollaries. Section 4 contains the proofs of these results. In the final Section 5 we discuss the case of random bonds; all results for the site problem have a natural counterpart in this situation.

2. DESCRIPTION OF THE MODELS

First we introduce the site model. Let $L = \mathbb{Z}^d$ denote the d-dimensional integer lattice. Spins and impurities are distributed on L by an *a priori* random mechanism which is given by a probability space $(\mathbb{S}, \mathcal{Q}, P)$; here $\mathbb{S}$ is the set of all subsets S of L, and $\mathcal{Q}$ is the σ-algebra on $\mathbb{S}$ which is generated by the mappings $\xi_x, x \in L$, where

$$\xi_x(S) = 1_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

If $\Lambda \in \mathbb{S}$ then we write ξ_Λ instead of $(\xi_x)_{x \in \Lambda}$. The probability measure P which governs the distribution of spins will often be assumed to be the Bernoulli measure P_p on $\mathbb{S}$ with probability $0 < p < 1$ for "success," i.e., P_p is the probability measure on $\mathbb{S}$ for which the variables ξ_x, $x \in L$, are independent and satisfy

$$P_p(\xi_x = 1) = p \tag{2.2}$$

Now suppose that $S \in \mathbb{S}$ is the set of all lattice sites which are occupied by a magnetic spin. Then we consider the set $\Omega_S = \{-1, 1\}^S$ of all spin configurations on S. Ω_S is endowed with the σ algebra $\mathcal{F}_S$ which is generated by the spin variables σ_x, $x \in S$; here

$$\sigma_x(\omega) = \omega_x \qquad \text{when} \quad \omega = (\omega_y)_{y \in S} \in \Omega_S \tag{2.3}$$

If $\Lambda \in \mathbb{S}$ then we use the abbreviation $\sigma_\Lambda = (\sigma_x)_{x \in \Lambda}$. The interaction of the spins is described by a potential

$$J : X \to J_X$$

which is a mapping from the set $\mathbb{S}_0$ of all nonempty finite subsets of L to the reals such that

$$\|J\|_x = \sum_{X \ni x} |J_X| < \infty \qquad \text{for all} \quad x \in L \tag{2.4}$$

Primarily we are interested in the case when J is a ferromagnetic pair potential with external field h, i.e., when

$$J_X = 0 \qquad \text{if} \quad |X| > 2 \tag{2.5}$$

$$J_X \geqslant 0 \qquad \text{if} \quad |X| = 2 \tag{2.6}$$

and

$$J_X = h \qquad \text{if} \quad |X| = 1 \tag{2.7}$$

($|X|$ is the cardinality of X.) J is said to be translationally invariant if

$$J_{X+x} = J_X \qquad \text{for all} \quad x \in L, X \in \mathbb{S}_0 \tag{2.8}$$

In particular, we will consider the Ising potential with external field h and coupling constant $\beta > 0$ which is given by

$$J_X = \begin{cases} h & \text{if} \quad |X| = 1 \\ \beta & \text{if} \quad |X| = 2 \quad \text{and} \quad \operatorname{diam} X = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

Now let J be given. The energy of a configuration $\omega_\Lambda = (\omega_x)_{x \in \Lambda} \in \Omega_\Lambda$ in a finite region $\Lambda \subset S$ with the boundary condition $\omega_{S \setminus \Lambda} = (\omega_x)_{x \in S \setminus \Lambda} \in \Omega_{S \setminus \Lambda}$ is given by the Hamiltonian

$$H_\Lambda^S(\omega_\Lambda \mid \omega_{S \setminus \Lambda}) = - \sum_{X \subset S \,:\, X \cap \Lambda \neq \varnothing} J_X \omega^X \tag{2.10}$$

where $\omega^X = \prod_{x \in X} \omega_x$. The Gibbs distribution in Λ with boundary condition $\zeta \in \Omega_{S \setminus \Lambda}$ is defined by

$$\gamma_\Lambda^S(\cdot \mid \zeta) = \exp\left[- H_\Lambda^S(\cdot \mid \zeta) \right] / Z_\Lambda^S(\zeta) \tag{2.11}$$

here

$$Z_\Lambda^S(\zeta) = \sum_{\alpha \in \Omega_\Lambda} \exp\left[- H_\Lambda^S(\alpha \mid \zeta) \right] \tag{2.12}$$

is the partition function. The object of our investigation is the set $\mathcal{G}_S(J)$ of all Gibbs measures (or equilibrium states) on Ω_S with respect to J. $\mathcal{G}_S(J)$ is the set of all probability measures μ_s on $(\Omega_S, \mathcal{F}_S)$ such that

$$\mu_S\left(\sigma_\Lambda = \omega \mid \mathcal{F}_{S \setminus \Lambda}^S\right) = \gamma_\Lambda^S(\omega \mid \sigma_{S \setminus \Lambda}) \, \mu_S - \text{a.s.} \tag{2.13}$$

for all $\Lambda \in \mathbb{S}_0$ with $\Lambda \subset S$ and all $\omega \in \Omega_\Lambda$; here we let $\mathcal{F}_{S \setminus \Lambda}^S$ denote the σ algebra on Ω_S which is generated by $\sigma_{S \setminus \Lambda}$.

The state space of the total system is $\tilde{\Omega} = \{-1, 0, 1\}^L$. $\tilde{\Omega}$ is endowed with the σ-algebras $\tilde{\mathcal{F}}_T$, $T \in \mathbb{S}$; $\tilde{\mathcal{F}}_T$ is generated by the projections $\tilde{\sigma}_x$, $x \in T$, which are given by $\tilde{\sigma}_x(\tilde{\omega}) = \tilde{\omega}_x$ if $\tilde{\omega} \in \tilde{\Omega}$. We put $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_L$. Each $\mu_S \in \mathcal{G}_S(J)$ has a unique extension $\tilde{\mu}_S$ to a measure on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ which is supported by

$$\{ \tilde{\sigma}_x = 0 \text{ iff } x \in L \setminus S \}$$

$\tilde{\mu}_S$ is an equilibrium state of the dilute spin system when the impurities are fixed on $L \setminus S$. In particular, the magnetization

$$\tilde{\mu}_S(\tilde{\sigma}_x) \equiv \int \tilde{\sigma}_x \, d\tilde{\mu}_S \tag{2.14}$$

at a lattice site $x \in L$ equals

$$\tilde{\mu}_S(\tilde{\sigma}_x) = \begin{cases} \mu_S(\sigma_x) \equiv \int \sigma_x \, d\mu_S & \text{if} \quad x \in S \\ 0 & \text{if} \quad x \in L \setminus S \end{cases} \tag{2.15}$$

Now let us suppose S is randomly chosen according to a law P on $(\mathbb{S}, \mathcal{Q})$. A conditional equilibrium state of the dilute spin system with a priori probability P is a probability measure $\tilde{\mu}$ on $(\tilde{\Omega}, \tilde{\mathfrak{F}})$ with the properties (i) $\pi(\tilde{\mu}) = P$, and (ii) $\tilde{\mu}(\cdot \mid \pi = S) \in \tilde{\mathcal{G}}_S(J)$ for P – a.a. S; here $\pi : \tilde{\Omega} \to \mathbb{S}$ is given by $\pi(\tilde{\omega}) = \{x \in L : \tilde{\omega}_x \neq 0\}$, and $\tilde{\mathcal{G}}_S(J) = \{\tilde{\mu}_S : \mu_S \in \mathcal{G}_S(J)\}$. We let $\tilde{\mathcal{G}}_P(J)$ denote the set of all these $\tilde{\mu}$. At the end of Section 4 we will prove

$$|\tilde{\mathcal{G}}_P(J)| > 1 \qquad \text{iff } P(S \in \mathbb{S} : |\mathcal{G}_S(J)| > 1) > 0 \tag{2.16}$$

This shows that the problem of phase transitions in dilute spin systems is, essentially, a problem concerning Gibbs measures on random subsets of L. This point of view will be adopted in Section 3.

Next we describe the bond model. Here at each lattice site $x \in L$ we have a spin taking the values ± 1. The spins interact via a potential J. However, there is an a priori random mechanism which removes the interaction between certain spins. (For example, one could think of random lattice defects or random impurities between the spins.) This random mechanism is described by a probability measure P on the set $\mathfrak{B}$ of all subsets B of $\mathbb{S}_0$; $\mathfrak{B}$ is endowed with the σ algebra which is generated by the mappings $\eta_X(B) = 1_B(X)$, $X \in \mathbb{S}_0$. Actually, only the restriction of P to the set of all subsets of $\{X \in \mathbb{S}_0 : J_X \neq 0\}$ is relevant. In particular, if J is the Ising potential (2.9) then we will use the same symbol $\mathfrak{B}$ to denote the set of all subsets of

$$E = \{e \subset L : \operatorname{diam} e = 1\} \tag{2.17}$$

the set of all edges between adjacent sites of L. The Bernoulli measure P_p with parameter $0 < p < 1$ on $\mathfrak{B} = \mathfrak{P}(E)$ is defined by the conditions (i) the variables η_e, $e \in E$, are independent, and (ii) $P_p(\eta_e = 1) = p$ for all $e \in E$. Now let J and $B \in \mathfrak{B}$ be given. For each $\Lambda \in \mathbb{S}_0$ and all spin configurations $\omega_\Lambda \in \Omega_\Lambda$ in Λ and boundary conditions $\omega_{L \setminus \Lambda} \in \Omega_{L \setminus \Lambda}$ we have the Hamiltonian

$$H_\Lambda^B(\omega_\Lambda \mid \omega_{L \setminus \Lambda}) = - \sum_{X \in B,\, X \cap \Lambda \neq \varnothing} J_X \omega^X \tag{2.18}$$

The Gibbs distribution in Λ with boundary condition $\zeta \in \Omega_{L \setminus \Lambda}$ and bond set B is given by

$$\gamma_\Lambda^B(\cdot \mid \zeta) = \exp\left[-H_\Lambda^B(\cdot \mid \zeta)\right] / Z_\Lambda^B(\zeta) \tag{2.19}$$

where

$$Z_\Lambda^B(\zeta) = \sum_{\omega \in \Omega_\Lambda} \exp\left[-H_\Lambda^B(\omega \mid \zeta)\right] \tag{2.20}$$

A probability measure μ_B on $(\Omega_L, \mathfrak{F}_L)$ is called a Gibbs measure with respect to B and J if for all $\Lambda \in \mathbb{S}_0$ and $\omega \in \Omega_\Lambda$

$$\mu_B\left(\sigma_\Lambda = \omega \mid \mathfrak{F}_{L \setminus \Lambda}^L\right) = \gamma_\Lambda^B(\omega \mid \sigma_{L \setminus \Lambda})\ \mu_B - \text{a.s.} \tag{2.21}$$

In Section 5 we will investigate the set $\mathcal{G}_B(J)$ of all Gibbs measures for B and J when B is randomly chosen. Just as in the site model we can also define conditional equilibrium states on $\mathfrak{B} \times \Omega_L$ with fixed or arbitrary *a priori* probability; the interested reader will find that these states admit a description which fits into the abstract theory of Gibbs random fields as presented in Ref. 11. However, this application does not seem to create interesting new problems. Therefore we prefer to stress the aspect of Gibbs measures on random graphs.

3. THE SITE MODEL: RESULTS

Here we state our results for the site model which was introduced in the previous section. We start with the observation that the structure of the set $\mathcal{G}_S(J)$ does not depend on the shape of S in finite regions.

Theorem 3.1. Let J be any potential and suppose that $S_1, S_2 \in \mathbb{S}$ have a finite symmetric difference $S_1 \triangle S_2$. Then there is a one-to-one correspondence φ between $\mathcal{G}_{S_1}(J)$ and $\mathcal{G}_{S_2}(J)$ which has the following property: If $\mu_{S_1} \in \mathcal{G}_{S_1}(J)$ then the measures $\tilde{\mu}_{S_1}$ and $\varphi(\tilde{\mu}_{S_1})$ are equivalent on $\tilde{\mathcal{F}}_{S_1 \cap S_2}$ and coincide on the tail field

$$\bigcap_{\Lambda \in \mathbb{S}_0} \tilde{\mathcal{F}}_{S_1 \setminus \Lambda} = \bigcap_{\Lambda \in \mathbb{S}_0} \tilde{\mathcal{F}}_{S_2 \setminus \Lambda}$$

The proof will be given in Section 4.

From now on we assume that J is a ferromagnetic pair potential with external field h. In this case it is well known (see Ruelle[12] and Lebowitz and Martin–Löf[13]) that for all $S \in \mathbb{S}$ there are two particular extreme elements μ_S^+ and μ_S^- of $\mathcal{G}_S(J)$ which are given by

$$\mu_S^+ = \lim_{\Lambda \uparrow S} \gamma_\Lambda^S(\cdot \mid +), \qquad \mu_S^- = \lim_{\Lambda \uparrow S} \gamma_\Lambda^S(\cdot \mid -) \tag{3.1}$$

(where $+$ and $-$ denote the configurations with the constant value $+1$ or -1, respectively). They satisfy

$$\begin{aligned} \mu_S^+ = \mu_S^- \qquad & \text{iff } |\mathcal{G}_S(J)| = 1 \\ \text{iff } \mu_S^+(\sigma_x) = \mu_S^-(\sigma_x) \qquad & \text{for all} \quad x \in S \end{aligned} \tag{3.2}$$

In particular, if $h = 0$ then

$$|\mathcal{G}_S(J)| > 1 \qquad \text{iff } \mu_S^+(\sigma_x) > 0 \qquad \text{for some} \quad x \in S. \tag{3.3}$$

(Usually these facts are only stated in the particular case $S = L$, but the proofs work when S is arbitrary.) As a by-product we obtain from (3.1) and (3.2) that the set

$$\mathbb{S}_1(J) = \{S \in \mathbb{S} : |\mathcal{G}_S(J)| = 1\} \tag{3.4}$$

is $\mathcal{A}$-measurable. Hence Theorem 3.1 implies that $\mathbb{S}_1(J)$ is even measurable with respect to the tail field $\mathcal{A}_\infty$; here

$$\mathcal{A}_\infty = \bigcap_{\Lambda \in \mathbb{S}_0} \mathcal{A}_{L \setminus \Lambda}$$

where $\mathcal{A}_{L\setminus\Lambda}$ is the σ-algebra which is generated by $\xi_{L\setminus\Lambda}$. Thus $P(\mathbb{S}_1(J)) = 0$ or 1 whenever P is a probability measure on $\mathbb{S}$ which is trivial on $\mathcal{A}_\infty$. According to Kolmogorov's 0-1 law this is particularly true when $P = P_p$ for some p. Moreover, if J is translationally invariant then so is $\mathbb{S}_1(J)$, and then we have $P(\mathbb{S}_1(J)) = 0$ or 1 whenever P is ergodic with respect to the translation group (in particular, when $P = P_p$ for some p). If $h \neq 0$ then $P(\mathbb{S}_1(J)) = 1$; this is the content of the following theorem which will be proved in Section 4.

Theorem 3.2. Suppose J is a translationally invariant ferromagnetic pair potential with external field $h \neq 0$, and P is a translationally invariant probability measure on $(\mathbb{S}, \mathcal{A})$. Then

$$|\mathcal{G}_S(J)| = 1$$

for P – almost all S.

Next we investigate the case when $h = 0$ and P is the Bernoulli measure P_p with respect to some p. Then we have

$$P_p(S \in \mathbb{S} : |\mathcal{G}_S(J)| > 1) = 1 \qquad \text{iff} \qquad \sup_{x \in L} \int P_p(dS) 1_S(x) \mu_S^+(\sigma_x) > 0 \tag{3.5}$$

i.e., spontaneous magnetization of the dilute spin system implies that, for almost all S, $\mathcal{G}_S(J)$ is not a singleton, and vice versa; this is a direct consequence of (3.3) and the 0-1 law above.

Now we ask for which concentrations p spontaneous magnetization occurs. This question has a complete answer when J is the Ising potential (2.9) with parameters $h = 0$, $\beta > 0$. This answer is possible because, for this J, there is much information available on the clusters of interacting spins. A finite sequence $(x_1, \ldots, x_n)$ of distinct (except that possibly $x_1 = x_n$) sites is called a path if, for all $1 \leqslant k < n$, x_k and x_{k+1} are adjacent, i.e., $\operatorname{diam}\{x_k, x_{k+1}\} = 1$. It is called a path from x to y (or, more generally, from S to T) if in addition $x_1 = x$, $x_n = y$ (or $x_1 \in S$, $x_n \in T$). A set $C \in \mathbb{S}$ is called connected if for all $x, y \in C$ there is a path from x to y in C. A maximal connected subset C of a given set $S \in \mathbb{S}$ is called a cluster of S.

If J is the Ising potential and $S \in \mathbb{S}$ then clearly the events on distinct clusters of S are independent for all $\mu_S \in \mathcal{G}_S(J)$. Moreover, the probability of events on a finite cluster of S is given by the Gibbs distribution on this cluster with free boundary condition; in particular, $\mu_S(\sigma_x) = 0$ for all x in a finite cluster of S, and if S consists only of finite clusters then $|\mathcal{G}_S(J)| = 1$. Therefore a dilute Ising ferromagnet does not exhibit spontaneous magnetization unless there is, with positive probability, an infinite cluster of spins. We let

$$\{\exists C^S_\infty\} \in \mathcal{A}_\infty \tag{3.6}$$

denote the set of all $S \in \mathbb{S}$ which contain an infinite cluster. It is well known[5, 7, 14] that there is a threshold value $0 < p_c(d) < 1$ (depending on the dimension d of L) such that

$$P_p(\exists C^S_\infty) = 0 \qquad \text{when} \quad p < p_c(d) \tag{3.7}$$

and

$$P_p(\exists C^S_\infty) = 1 \qquad \text{when} \quad p > p_c(d) \tag{3.8}$$

$p_c(d)$ is called the critical probability. [Monte Carlo estimates[7, 14] give $p_c(2) = 0.590$ and $p_c(3) = 0.307$.] Obviously, (3.8) implies that for $p > p_c(d)$ we also have

$$P_p(x \in C^S_\infty) > 0 \qquad \text{for all} \quad x \in L \tag{3.9}$$

here $\{x \in C^S_\infty\}$ is the event that x belongs to an infinite cluster of S. Moreover, in dimension $d = 2$ it is known[15] that for all $p > p_c(2)$ and P_p – almost all S there is only one infinite cluster in S; this cluster will be denoted by C^S_∞.

From (3.7) we immediately obtain the following

Remark 3.1. Let $h = 0$, $\beta > 0$ and J be given by (2.9). Then for all $p < p_c(d)$ and P_p – almost all S we have

$$|\mathcal{G}_S(J)| = 1$$

The next theorem is our main result.

Theorem 3.3. Suppose J is the Ising potential (2.9) with parameters $h = 0$, $\beta > 0$. Let $d = 2$ and $p > p_c(2)$. Then

$$P_p\Big(S \in \mathbb{S}:\ \lim_{\beta\to\infty} \mu_S^+(\sigma_x) = 1 \qquad \text{for all} \quad x \in C^S_\infty\Big) = 1 \tag{3.10}$$

In particular,

$$\lim_{\beta\to\infty} \int P_p(dS)\tilde{\mu}_S^+(\tilde{\sigma}_x) = P_p(x \in C^S_\infty) > 0 \tag{3.11}$$

and there is a finite critical value $\beta_c(p)$ such that

$$P_p(S \in \mathbb{S} : |\mathcal{G}_S(J)| = 1) = 1 \qquad \text{when} \quad \beta < \beta_c(p) \tag{3.12}$$

and

$$P_p(S \in \mathbb{S} : |\mathcal{G}_S(J)| > 1) = 1 \qquad \text{when} \quad \beta > \beta_c(p) \tag{3.13}$$

Moreover, for $\beta > \beta_c(p)$ and P_p – almost all S we have

$$\mu_S^+(\sigma_x) > 0 \qquad \text{for all} \quad x \in C_\infty^S \tag{3.14}$$

but

$$\inf_{x \in C_\infty^S} \mu_S^+(\sigma_x) = 0 \tag{3.15}$$

Let us note that because of (3.15), the limit in (3.10) is almost surely not uniform in x. On the other hand, the maximal magnetization in the infinite cluster

$$\sup_{x \in C_\infty^S} \mu_S^+(\sigma_x)$$

is an almost surely constant function of S; this is because this function is invariant under translations and P_p is ergodic.

Now we discuss some extensions of Theorem 3.3. The first observation is that also the dilute Ising antiferromagnet shows a phase transition; this follows from (3.13) by a spin reversal on the even sublattice of L. Further extensions can be obtained using the second Griffiths inequality (see, for instance, Sylvester[16]). A standard application of this inequality gives that, for all $x \in \Lambda \in \mathbb{S}_0$, the expected magnetization

$$m_\Lambda(x,S,J) = 1_S(x)\gamma_{\Lambda\cap S}^S(\sigma_x \mid +)$$

is an increasing function of S and J provided J is ferromagnetic. Thus (3.1) and (3.3) imply the following.

Remark 3.2. Let $S_1, S_2 \in \mathbb{S}$ and suppose that J_1, J_2 are ferromagnetic pair potentials with zero external fields. Assume $S_1 \subset S_2$ and $J_1 \leqslant J_2$. Then

$$|\mathcal{G}_{S_1}(J_1)| > 1 \qquad \text{implies} \quad |\mathcal{G}_{S_2}(J_2)| > 1$$

Combining (3.13) and Remark 3.2 we obtain spontaneous magnetization of dilute ferromagnets with long-range interactions in arbitrary dimension $d \geqslant 2$. (The extension to higher dimensions uses the fact that a two-dimensional system can be considered as a layer in a d-dimensional system without interaction between parallel layers.) We put

$$m(x,S,J) = \lim_{\Lambda \uparrow S} m_\Lambda(x,S,J) = \tilde{\mu}_S^+(\tilde{\sigma}_x)$$

when $\mu_S^+ \in \mathcal{G}_S(J)$.

Corollary 3.1. Let $d \geqslant 2$, $p > p_c(2)$ and suppose that J is any ferromagnetic pair potential with zero external field. Then

$$\beta_c(J,p) = \sup\left\{ \beta \geqslant 0 \colon \int P_p(dS)m(x,S,\beta J) = 0 \right\}$$

is finite and a decreasing function of J; moreover

$$P_p(S: |\mathcal{G}_S(\beta J)| = 1) = 1 \qquad \text{when} \quad \beta < \beta_c(J, p)$$

and

$$P_p(S: |\mathcal{G}_S(\beta J)| > 1) = 1 \qquad \text{when} \quad \beta > \beta_c(J, p)$$

Next we use the monotonicity of the expected magnetization to replace the Bernoulli measures P_p by measures with nontrivial correlations. This gives us spontaneous magnetization of dilute systems in which the positions of spins and impurities are in equilibrium with respect to certain nontrivial interactions. We introduce an ordering between probability measures on $(\mathbb{S}, \mathcal{Q})$ by writing $P \prec Q$ if

$$\int f\,dP \leqslant \int f\,dQ$$

for all increasing real functions f of the form $f = g(\xi_\Lambda)$ with $\Lambda \in \mathbb{S}_0$. Examples for measures P, Q with $P \prec Q$ are provided by the FKG Holley[17] inequality. For instance, let

$$\Phi : \mathbb{S}_0 \to \mathbb{R}$$

be a function which satisfies

$$\Phi(X) \geqslant 0 \qquad \text{when} \quad |X| \geqslant 2$$

and

$$\sum_{X \ni x} |\Phi(X)| < \infty \qquad \text{for all} \quad x \in L$$

define the set $\mathcal{G}(\Phi)$ of Gibbs measures for Φ on $(\mathbb{S}, \mathcal{Q})$ as usual by means of the Hamiltonians

$$H_\Lambda(S \mid T) = - \sum_{X \subset S \cup T,\, X \cap \Lambda \neq \Phi} \Phi(X)$$

(where $\Lambda \in \mathbb{S}_0$, $S \subset \Lambda$, $T \subset L \backslash \Lambda$). Let $0 < p < 1$ and suppose

$$\log \frac{p}{1-p} \leqslant \inf_{x \in L} \Phi(\{x\})$$

Then

$$P_p \prec Q \qquad \text{for all} \quad Q \in \mathcal{G}(\Phi)$$

In particular, if $p_1 < p_2$ then $P_{p_1} \prec P_{p_2}$.

Inserting the increasing functions $m_\Lambda(x, \cdot, J)$ into the definition of "$\prec$" and letting Λ tend to infinity we obtain the following corollary.

Corollary 3.2. Let $d \geqslant 2$. Assume J is a ferromagnetic pair potential with zero exernal field and P is a probability measure on $(\mathbb{S}, \mathcal{Q})$ such that, for some $p > p_c(2)$, $P_p \prec P$. Then for sufficiently large β we have

$$\inf_{x \in L} \int P(dS) m(x, S, \beta J) > 0$$

and

$$P(S \in \mathbb{S} : |\mathcal{G}_S(\beta J)| > 1) > 0$$

In particular, $\beta_c(J, p)$ is a decreasing function of p.

It should be mentioned that the monotonicity properties of $m(x, S, J)$ and $\beta_c(J, p)$ were already observed by Griffiths and Lebowitz.[6] Further extensions of Corollary 3.2 can be obtained using the following device: Suppose P has a representation

$$P = \int W(dQ) Q$$

by a probability measure W on the set of all probability measures on $\mathbb{S}$; assume

$$W(Q: P_p \prec Q \qquad \text{for some} \quad p > p_c(2)) > 0$$

Then the statement of Corollary 3.2 remains true. For instance, this remark applies to symmetric measures P with $\int \xi_x \, dP > p_c(2)$ or, more generally, to certain canonical Gibbs measures as studied in Ref. 18.

We conclude this section with some remarks. Theorem 3.3 might be considered as a very first step of the program of describing all Markov random fields on a countably infinite graph or, at least, on all subgraphs S of L. We have seen that the phenomenon of nonuniqueness is not restricted to the well-studied case $S = L$, but occurs whenever S contains a typical realization of P_p for some $p > p_c(2)$; this is a partial answer to a question raised by Dobrushin.[19] However, the problem of determining all extreme points of $\mathcal{G}_S(J)$ remains open. In particular, it is not obvious whether in dimension $d = 2$

$$\mathcal{G}_S(J) = [\, \mu_S^-, \mu_S^+ \,]$$

when $S \subset L$ and J is the Ising potential; it is a famous result of Aizenman[20] and Higuchi[21] that this holds when $S = L$, but their proofs need all symmetries of L which apparently cannot be replaced by the symmetries of P_p. Thus even for "typical" S the problem is unsolved.

However, one fact carries over to arbitrary S. In the case $S = L$ it was shown by Coniglio *et al.*[22] and Russo[23] that spontaneous magnetization gives rise to the existence of an infinite $(+)$ cluster, i.e., of an infinite cluster of $\{x \in L: \tilde{\sigma}_x = 1\}$. Their arguments can be extended to cover the general case. Therefore we may state [letting $\{\exists C_\infty^+\}$ denote the event in $\tilde{\Omega}$ that an infinite $(+)$ cluster exists] the following.

Remark 3.3. Let $d \geqslant 2$ and $S \in \mathbb{S}$ and suppose J is of the form (2.9). Assume that

$$|\mathcal{G}_S(J)| > 1$$

for $h = 0$ and some $\beta_0 > 0$. Then

$$\tilde{\mu}_S^+ (\exists C_\infty^+) = 1$$

whenever $h \geqslant 0$ and $\beta \geqslant \beta_0 - h/2d$.

This shows that a randomly dilute Ising ferromagnet with parameters $p > p_c(2)$ and $\beta > \beta_c(p) - h/2d$ almost surely contains an infinite $(+)$ cluster.

4. THE SITE MODEL: PROOFS

We start with the proof of Theorem 3.3. Thus we assume $d = 2$ and suppose that J is the Ising potential with $h = 0$ and some given $\beta > 0$. The main idea is a generalized version of the well-known Peierls argument. So we have to define contours. Let $S \in \mathbb{S}$ be fixed.

First we introduce the dual lattice

$$L' = L + \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

and the set

$$E' = \{e' \subset L' : \operatorname{diam} e' = 1\}$$

of edges between adjacent sites of L'. Each edge $e' = \{x', y'\} \in E'$ is visualized by the line in $\mathbb{R}^2$ which connects x' and y'. This line crosses a unique line which characterizes an edge $e(e') \in E$. Conversely, $e'(e)$ is defined by $e(e'(e)) = e$, $e \in E$. We say that two vertices $x \in L$ and $x' \in L'$ are contiguous if $x' = x + (\pm\frac{1}{2}, \pm\frac{1}{2})$ or, equivalently, if $x' \in e'(e)$ for some $e \in E$ with $x \in e$. Two sets $D \in L$ and $D' \subset L'$ are said to be contiguous if there is a pair x, x' of contiguous vertices with $x \in D$, $x' \in D'$.

A finite sequence $c = (x_1', \ldots, x_n')$ of distinct (except that possibly $x_1' = x_n'$) sites of L' is called an S polygon if, for all $1 \leqslant k < n$, $\{x_k', x_{k+1}'\} \in E'$ and $e(\{x_k', x_{k+1}'\}) \subset S$. c is called an S polygon from $x \in L$ to $y \in L$ if x_1' is contiguous to x and x_n' is contiguous to y, and an S polygon from $D_1 \subset L$ to $D_2 \subset L$ if c is an S polygon from a site of D_1 to a site of D_2. Finally, an S polygon c with $x_1' = x_n'$ is called an $(S, 0)$ contour.

To define (S, n) contours for $n \geqslant 1$ we need the notion of a *cluster. Two sites $x = (x^1, x^2)$ and $y = (y^1, y^2)$ of L are *adjacent if $|x^1 - y^1| \leqslant 1$ and $|x^2 - y^2| \leqslant 1$. A set $D \subset L$ is called *connected if for all $x, y \in D$ there is a finite sequence $(x_1, \ldots, x_n)$ in D such that $x = x_1$, $y = x_n$ and, for all $1 \leqslant k < n$, x_k and x_{k+1} are *adjacent. A maximal *connected subset of $L \backslash S$ is called a *cluster of $L \backslash S$ or shorter an $\bar{S}$ cluster. An alternating sequence $g = (c_1, D_1, \ldots, c_n, D_n)$ of mutually disjoint S polygons

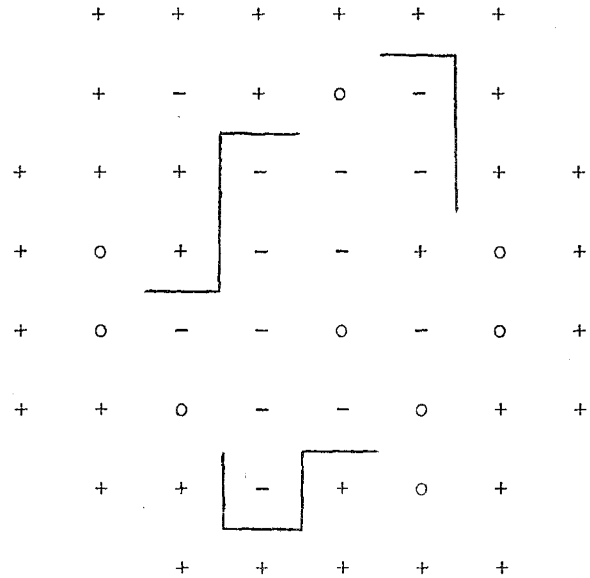

Fig. 1. A configuration ω of spins (+ or −) on a set S and impurity atoms (0) on $L \backslash S$. The impurity sites fall into three *clusters which, together with three S polygons (solid lines), form an $(S, 3)$ contour g. g is realized by ω, and the circuit at the boundary of the figure defines an S region containing g.

$c_1, \ldots, c_n$ and pairwise distinct finite $\bar{S}$ clusters $D_1, \ldots, D_n$ will be called an (S, n) contour if, for all $1 \leqslant k \leqslant n$, c_k is an S polygon from D_{k-1} to D_k (where $D_0 \equiv D_n$); see Fig. 1.

Now let g be any S contour, i.e., an (S, n) contour for some $n \geqslant 0$. We say g is an (S, n) contour in $\Lambda \in \mathbb{S}_0$ if all $\bar{S}$ clusters in g are subsets of Λ and every vertex of each S polygon in g is contiguous to a site in Λ. g is said to surround a site $x \in L$ if for each Λ containing g and each path w from x to $L \backslash \Lambda$ the following holds: Either w contains sites of the $\bar{S}$ clusters in g, or w contains an edge which crosses an S polygon in g. Finally, we say that g is realized in a configuration $\omega \in \Omega_S$ if $\omega_{y_1} \neq \omega_{y_2}$ whenever $\{y, y_2\} = e(e')$ for some edge e' in any S polygon of g.

Next we introduce particular regions in L. A path $(x_1, \ldots, x_n)$ in L is called a circuit if $x_1 = x_n$. Each circuit has an interior in the obvious sense (such that the circuit and its interior are disjoint). A set $\Lambda \in \mathbb{S}_0$ is called an S region if Λ is the interior of a circuit in S.

Lemma 4.1. Let $S \in \mathbb{S}$. Suppose x belongs to an infinite cluster of S and Λ is an S region containing x. Assume that $\omega \in \Omega_S$ with $\omega_x = -1$ and $\omega_{S \backslash \Lambda} \equiv 1$. Then there is an S contour in Λ which surrounds x and is realized in ω.

Proof. By assumption there is a circuit $\partial\Lambda$ in S such that Λ is the interior of $\partial\Lambda$. Moreover, there exists a cluster C of $\{y \in S : \omega_y = -1\}$ with $x \in C \subset S \cap \Lambda$. We let $\partial_S C$ denote the outer boundary of C in S, i.e., the set of all vertices in the infinite cluster of $C_\infty^S \setminus C$ which are adjacent to a site of C; here C_∞^S is the infinite cluster of S containing x. Clearly, $\partial\Lambda \subset C_\infty^S$.

First we observe that each path from x to $\partial\Lambda$ either meets $\Lambda \setminus S$ or crosses an edge of

$$B' = \bigcup\{e'(\{y_1, y_2\}) : \{y_1, y_2\} \in E, y_1 \in C, y_2 \in \partial_S C\}$$

Indeed, assume the path is contained in S; then the "last exit of the path from C" defines an edge $\{y_1, y_2\} \in E$ with $y_1 \in C, y_2 \in \partial_S C$. Moreover, it is easily verified that if $x' \in B'$ then B' contains at most two sites which are adjacent to x'. Hence B' can be decomposed into finitely many S polygons $c_1, \ldots, c_n$. We need to show that these S polygons are the S polygons of an S contour. Then, by construction, this S contour will be realized in ω.

If $n = 1$ and c_1 is an $(S,0)$ contour then we are done. In the opposite case, each S polygon in B' has precisely two end points. Each such end point is contiguous to an $\bar{S}$ cluster. These $\bar{S}$ clusters are denoted by $D_1, \ldots, D_m$. Since Λ is an S region, $D_1 \cup \cdots \cup D_m \subset \Lambda$.

Next we consider a fixed S polygon c_k in B'. Its two end points are contiguous to two distinct sites $x \in D_i$ and $y \in D_j$, say. If $D_i = D_j$ then there is a path in $\Lambda \setminus S$ from x to y; since c_k is contiguous to C_∞^S we necessarily have $m = n = 1$, and (c_1, D_1) is an $(S, 1)$ contour surrounding x.

Now let $m > 1$. Then from the preceding we know that each S polygon in B' is an S polygon from some D_j to a different D_k. Conversely, each D_k is contiguous to precisely two distinct S polygons in B'. For suppose first that D_k is contiguous to three distinct S polygons in B'. Then we can find three distinct paths from x to $\partial\Lambda$, each path crossing one of these S polygons. The three paths define three regions in Λ, two of which must contain sites of D_k. This is impossible since D_k is *connected. On the other hand, there exist two paths in C_∞^S from x to a vertex of $\partial\Lambda$ such that D_k but no other D_j is situated between them. At their last exit from C, both paths cross S polygons in B'. These S polygons are contiguous to D_k and distinct since otherwise $m = 1$. Now it is immediate to conclude that $m = n$, and (up to a proper numbering) $(c_1, D_1, \ldots, c_n, D_n)$ is an (S, n) contour surrounding x. ■

Now we turn to the proof of Theorem 3.3. We start from the estimate

$$\gamma_{\Lambda \cap S}^S(\sigma_x = -1 \mid +) \leqslant \sum_{g = g_{S,\Lambda,x}} \gamma_{\Lambda \cap S}^S(\omega : g \text{ is realized in } \omega \mid +) \quad (4.1)$$

which is valid under the hypotheses of Lemma 4.1; the sum extends over all

S contours g in Λ which surround x. The right-hand side of (4.1) has an upper bound in terms of the length $l(g)$ of g; $l(g)$ is the total number of edges in all S polygons of g. This upper bound is obtained by a straightforward modification of the well-known Peierls estimate (which is based on a spin reversal at all sites of $\Lambda \cap S$ which are surrounded by g; see Griffiths.[24]

Lemma 4.2. Let J be given by (2.9) with $h = 0$ and $\beta > 0$. Let $S \in \mathbb{S}$ and suppose Λ is an S region and g an S contour in Λ. Then

$$\gamma^S_{\Lambda \cap S}(\omega : g \text{ is realized in } \omega \,|\, +) \leqslant r^{l(g)}$$

where $r = e^{-2\beta}$.

Now let $p > p_c(2)$ and $P = P_p$ be the Bernoulli measure on $\mathbb{S}$ with respect to p. It is known (see the proof of Theorem 2 of Russo[15]; compare also the proof of Lemma 4.4 below) that for P – almost all S the following holds: Every $\Lambda \in \mathbb{S}_0$ is contained in the interior of a circuit in S, i.e., there is an increasing sequence Λ_n of S regions with $\cup \Lambda_n = L$. (In particular, this shows that almost surely there is only one infinite cluster in S.) Combining this and (3.1) we obtain from (4.1) and Lemma 4.2 that

$$1_{\{x \in C^S_\infty\}}\, \mu^+_S\,(\sigma_x = -1) \leqslant \sum_{g = g_{S,x}} r^{l(g)} \tag{4.2}$$

for all $x \in L$ and P – almost all S; the sum extends over all S contours g surrounding x. We estimate the expectation of this sum.

Fix any $x \in L$. For $S \in \mathbb{S}$ we let

$$N_S(0,l)$$

denote the total number of $(S,0)$ contours of length l which surround x. Similarly, for $n \geqslant 1$ we let

$$N_S(n, l_1, \ldots, l_n)$$

denote the number of all (S,n) contours $(c_1, D_1, \ldots, c_n, D_n)$ which surround x and are such that, for all $1 \leqslant k \leqslant n$, c_k contains l_k edges. Then (4.2) gives

$$\begin{aligned}
&\int_{\{x \in C^S_\infty\}} P(dS)\mu^+_S\,(\sigma_x = -1) \\
&\quad \leqslant \int P(dS)\Bigg[\sum_{l \geqslant 1} r^l N_S(0,l) \\
&\qquad + \sum_{n \geqslant 1}\ \sum_{l_1, \ldots, l_n \geqslant 1} r^{l_1 + \cdots + l_n} N_S(n, l_1, \ldots, l_n)\Bigg]
\end{aligned} \tag{4.3}$$

We need to show that the right-hand side of (4.3) tends to zero as r tends to zero. Indeed, this would imply

$$\lim_{\beta\to\infty}\int P(dS)\tilde{\mu}_S^+(\tilde{\sigma}_x)$$
$$= \lim_{\beta\to\infty}\int_{\{x\in C_\infty^S\}} P(dS)\left[1-2\mu_S^+(\sigma_x=-1)\right] = P\left(x\in C_\infty^S\right) \quad (4.4)$$

this is the basic assertion (3.11) of Theorem 3.3. Clearly,

$$\sum_{l\geq 1} N_S(0,l)r^l \leq \sum_{l\geq 1} l3^l r^l \quad (4.5)$$

and the right-hand side vanishes in the limit $r\to 0$.

Next we fix $n\geq 1$, $l_1,\ldots,l_n\geq 1$ and consider

$$\int P(dS)N_S(n,l_1,\ldots,l_n)$$

let us write

$$N_S(n,l_1,\ldots,l_n) \leq \sum_{k_1,\ldots,k_n\geq 0}\sum{}' I_S(y_1,\ldots,y_n,x_2,\ldots,x_n,k_n) \quad (4.6)$$

Here the primed sum extends over all $x_1,\ldots,x_n,y_1,\ldots,y_n\in L$ and all L polygons $c_1,\ldots,c_n$ with the following properties:

(i) the L^1-type distance $|x_1-x|$ of x_1 and x satisfies $|x_1-x|\leq l_1+\cdots+l_n+k_1+\cdots+k_n+2n$;
(ii) for all $1\leq i\leq n$, c_i is an L polygon of length l_i from x_i to y_i;
(iii) for all $1\leq i<n$, $|x_{i+1}-y_i|=k_i$.

The function I_S in (4.6) is defined as follows: $I_S=1$ if there are mutually distinct $\bar{S}$ clusters $D_1,\ldots,D_n$ such that

(I) for all $1\leq i<n$, $\{y_i,x_{i+1}\}\subset D_i$
(II) $y_n\in D_n$, and D_n contains a site at distance k_n from y_n.

Otherwise we put $I_S=0$. In order to estimate the expectation of I_S we introduce random sets

$$\Delta(y):\mathbb{S}\to\mathbb{S}_0\cup\{\emptyset\}$$

as follows: If $\xi_y=0$ then $\Delta(y)$ is the *cluster of $\{k\in L:\xi_k=0\}$ which contains y; otherwise $\Delta(y)=\emptyset$. Accordingly, we let $\{x\in\Delta(y)\}$ denote the event that x belongs to $\Delta(y)$; we write $\{\delta(y)\geq k\}$ to denote the event that $\Delta(y)$ contains a vertex at distance k from y.

Lemma 4.3. Let $n\geq 1$, $k_n\geq 0$ and $y_1,\ldots,y_n$, $x_2,\ldots,x_n\in L$. Then

$$\int P(dS)I_S(y_1,\ldots,y_n,x_2,\ldots,x_n,k_n)$$
$$\leq P(x_2\in\Delta(y_1))\cdots P(x_n\in\Delta(y_{n-1}))P(\delta(y_n)\geq k_n).$$

The proof is postponed until after Lemma 4.4. Combining (4.6) and Lemma 4.3 we obtain

$$\begin{aligned}
&\int P(dS)N_S(n,l_1,\ldots,l_n)\\
&\quad\leqslant \sum_{k_1,\ldots,k_n\geqslant 0} \sum{}' P(x_2\in\Delta(y_1))\cdots P(x_n\in\Delta(y_{n-1}))P(\delta(y_n)\geqslant k_n)\\
&\quad\leqslant \sum_{k_1,\ldots,k_n\geqslant 0} 4(l_1+\cdots+l_n+k_1+\cdots+k_n+2n)^2\\
&\qquad\times\left\{\prod_{i=1}^{n}(8\cdot 3^{l_i-1}\cdot 2)\right\}\left\{\prod_{i=1}^{n-1}(4k_i+1)\right\}\left\{\prod_{i=1}^{n}P(\delta(0)\geqslant k_i)\right\}\\
&\quad\leqslant (22)^n \sum_{k_1,\ldots,k_n\geqslant 0}\left[l_1+\cdots+l_n+(k_1+1)+\cdots+(k_n+1)+n\right]^2\\
&\qquad\times 3^{l_1+\cdots+l_n}\prod_{i=1}^{n}\{(k_i+1)P(\delta(0)\geqslant k_i)\}
\end{aligned}\tag{4.7}$$

Now we use the inequality

$$(j_1+\cdots+j_m)^2\leqslant m^2j_1^2\cdots j_m^2$$

which is valid whenever $j_1,\ldots,j_m\geqslant 1$. We obtain

$$\int P(dS)N_S(n,l_1,\ldots,l_n)\leqslant 4(n+1)^4K^n\prod_{i=1}^{n}(l_i^2 3^{l_i})\tag{4.8}$$

where

$$K=22\sum_{k\geqslant 0}(k+1)^3P(\delta(0)\geqslant k)\tag{4.9}$$

Therefore the second term on the right-hand side of (4.3) has the upper bound

$$4\sum_{n\geqslant 1}(n+1)^4\left\{K\sum_{l\geqslant 1}l^2(3r)^l\right\}^n\tag{4.10}$$

Suppose we knew K was finite. Then (4.10) tends to zero as r tends to zero. Combined with (4.5) this gives (4.4), and the proof of (3.11) is complete. The assertion $K<\infty$ follows from the next lemma.

Lemma 4.4. Suppose $p>p_c(2)$ and p is the Bernoulli measure on $\mathbb{S}$ for p. Then for all $m\geqslant 0$ we have

$$\sum_{k\geqslant 1}k^mP(\delta(0)\geqslant k)<\infty$$

Proof. This result is implicitly contained in the proof of Theorem 2 of Russo,[15] which is based on the ideas of Seymour and Welsh.[25] Let

$$\Lambda(j)=\{x=(x^1,x^2)\in L:|x^1|\leqslant j,|x^2|\leqslant j\}$$

Choose any $0 < \epsilon < 3^{-m-1}$. Russo proves that there is a number $n = n(p)$ such that, for all $j \geqslant 1$, the set

$$\{x \in \Lambda(3^j n)\setminus\Lambda(3^{j-1}n) : \xi_x = 0\}$$

contains a circuit surrounding the origin with probability larger than $1 - \epsilon$. This implies that

$$P(\Delta(0)\setminus\Lambda(3^j n) \neq \emptyset) \leqslant \epsilon^j$$

for all $j \geqslant 1$. Now we write

$$\sum_{k\geqslant 1} k^m P(\delta(0) \geqslant k) \leqslant 2^{m+1} \sum_{k\geqslant 1} k^m P(\delta(0) \geqslant 2k-1)$$

$$\leqslant (2n)^{m+1} + 2^{m+1} \sum_{j\geqslant 0} \sum_{k=3^j n+1}^{3^{j+1}n} k^m P(\Delta(0)\setminus\Lambda(k-1) \neq \emptyset)$$

The last term is dominated by the convergent series

$$(6n)^{m+1} \sum_{j=0}^{\infty} (3^{m+1}\epsilon)^j$$

This proves the lemma. ■

Proof of Lemma 4.3. By definition we have

$$\int P(dS) I_S(y_1, \ldots, y_n, x_2, \ldots, x_n, k_n)$$

$$= \sum_{D_1, \ldots, D_{n-1}} P(\Delta(y_1) = D_1, \ldots, \Delta(y_{n-1}) = D_{n-1}, \delta(y_n) \geqslant k_n) \tag{4.11}$$

the sum extends over all *connected finite sets $D, \ldots, D_{n-1}$ with the following properties: The sets $D_i \cup D_j$ $(i \neq j)$ are not *connected, and $\{y_i, x_{i+1}\} \subset D_i$, $y_n \notin D_i$ for all i. The right-hand side of (4.11) can be written as

$$\sum_{D_1} P(\Delta(y_1) = D_1) \sum_{D_2} P(\Delta(y_2) = D_2 \,|\, \Delta(y_1) = D_1)$$

$$\cdots \sum_{D_{n-1}} P(\Delta(y_{n-1}) = D_{n-1} \,|\, \Delta(y_1) = D_1, \ldots, \Delta(y_{n-2}) = D_{n-2})$$

$$\times P(\delta(y_n) \geqslant k_n \,|\, \Delta(y_1) = D_1, \ldots, \Delta(y_{n-1}) = D_{n-1})$$

If $D \subset L$ then we let $\partial^* D$ denote the set of all vertices of $L\setminus D$ which are *adjacent to a site of D. Then the last factor equals

$$P(\delta(y_n) \geqslant k_n \,|\, \xi_{D_1 \cup \cdots \cup D_{n-1}} \equiv 0, \xi_{\partial^*(D_1 \cup \cdots \cup D_{n-1})} \equiv 1)$$

In this expression the event $\{\delta(y_n) \geqslant k_n\}$ can be replaced by the subevent that y_n belongs to a *cluster of

$$\{z \notin (D_1 \cup \cdots \cup D_{n-1}) \cup \partial^*(D_1 \cup \cdots \cup D_{n-1}) : \xi_z = 0\}$$

which also contains a site at distance k_n from y_n. This event is independent of the event in the condition. Therefore the conditional probability above is less than $P(\delta(y_n) \geqslant k_n)$. A similar argument shows that for all $1 < m < n$

$$\sum_{D_m} P(\Delta(y_m) = D_m \,|\, \Delta(y_1) = D_1, \ldots, \Delta(y_{m-1}) = D_{m-1})$$

$$= P(x_{m+1} \in \Delta(y_m) \,|\, \Delta(y_1) = D_1, \ldots, \Delta(y_{m-1}) = D_{m-1})$$

$$\leqslant P(x_{m+1} \in \Delta(y_m))$$

Now the conclusion of Lemma 4.3 is obvious. ■

Having completed the proof of (3.11) we turn to the proofs of the further assertions of Theorem 3.3.

Proof of (3.10), (3.12), and (3.13). Because of Griffiths' second inequality[16] $\mu_S^+(\sigma_x)$ is an increasing function of β; this was already observed in Remark 3.2. Therefore (3.12) and (3.13) follow from (3.5), (3.11), and the 0–1 laws which were discussed in the paragraph below (3.4). Moreover,

$$\lim_{\beta\to\infty} \mu_S^+(\sigma_x)$$

exists for all $x \in S \in \mathbb{S}$; thus (3.11) and the dominated convergence theorem imply

$$\int_{\{x \in C_\infty^S\}} P(dS)\left[1 - \lim_{\beta\to\infty} \mu_S^+(\sigma_x)\right] = 0$$

for all $x \in L$. This gives

$$P\left(S : \lim_{\beta\to\infty} \mu_S^+(\sigma_x) < 1 \quad \text{for some} \quad x \in C_\infty^S\right) = 0$$

and hence (3.10). ■

Proof of (3.14). From (3.3) and (3.13) we know that for $\beta > \beta_c(p)$

$$P\left(S : \sup_{x \in C_\infty^S} \mu_S^+(\sigma_x) > 0\right) = 1$$

Therefore it suffices to prove that if $S \in \mathbb{S}$ and $x, y \in S$ are adjacent then

$$\mu_S^+(\sigma_x) = 0 \qquad \text{implies} \quad \mu_S^+(\sigma_y) = 0$$

To show this we fix an ϵ with $0 < \epsilon \leqslant \beta$ and let ν^ϵ denote the measure $\mu_S^+ \in \mathcal{G}_S(J^\epsilon)$, where

$$J_X^\epsilon = \begin{cases} J_X & \text{when } X \neq \{x, y\} \\ \beta - \epsilon & \text{when } X = \{x, y\} \end{cases}$$

J^ϵ is a ferromagnetic pair potential. Clearly,

$$\mu_S^+(\sigma_x) = \nu^\epsilon(\sigma_x \exp(\epsilon\sigma_x\sigma_y))/\nu^\epsilon(\exp(\epsilon\sigma_x\sigma_y))$$

Using the identity

$$\exp(\epsilon\sigma_x\sigma_y) = \cosh\epsilon + \sigma_x\sigma_y \sinh\epsilon$$

we obtain

$$\cosh\epsilon\, \nu^{\epsilon}(\sigma_x) + \sinh\epsilon\, \nu^{\epsilon}(\sigma_y) = 0$$

Griffiths' two inequalities give

$$0 \leqslant \nu^{\epsilon}(\sigma_x) \leqslant \mu_S^+(\sigma_x) = 0$$

Hence $\nu^{\epsilon}(\sigma_y) = 0$ and therefore

$$\mu_S^+(\sigma_y) = \lim_{\epsilon\to 0} \nu^{\epsilon}(\sigma_y) = 0 \quad \blacksquare$$

Proof of (3.15). We put

$$Q = \{x = (x^1, x^2) \in L : x^1 \geqslant 0, x^2 \geqslant 0\}$$

and let

$$\{0 \in C_{\infty}^{S\cap Q}\}$$

denote the event that the origin belongs to an infinite cluster of

$$\{x \in Q : \xi_x = 1\}$$

Theorem 2 of Russo,[15] combined with a straightforward adaptation of Theorem 3.2 of Smythe and Wierman[25] to the present context, gives

$$P(0 \in C_{\infty}^{S\cap Q}) > 0$$

Next we fix an integer $N \geqslant 1$ and consider the event

$$A = \{\xi_\Lambda \equiv 1, \xi_\Delta \equiv 0\}$$

where

$$\Lambda = \{(k,0) : -N \leqslant k < 0\}$$

and

$$\Delta = \{(k, \pm 1) : -N \leqslant k < 0\} \cup \{(-N-1, 0)\}$$

Obviously we have $P(A) > 0$; thus

$$P(A \cap \{0 \in C_{\infty}^{S\cap Q}\}) > 0$$

since Q and $\Lambda \cup \Delta$ are disjoint.

Now let $S \in A \cap \{0 \in C_{\infty}^{S\cap Q}\}$. Then Λ is contained in an infinite cluster of S, and $S \cap \Delta = \emptyset$. For fixed $x = (k,0) \in \Lambda$ we put $y = (k+1, 0)$

and $V = \{(j,0) : -N \leqslant j \leqslant k\}$. Then

$$\mu_S^+(\sigma_x) = \int_{\{\sigma_y=1\}} d\mu_S^+ \gamma_V^S(\sigma_x \mid +) + \int_{\{\sigma_y=-1\}} d\mu_S^+ \gamma_V^S(\sigma_x \mid -)$$
$$= \gamma_V^S(\sigma_x \mid +)\, \mu_X^+(\sigma_y) \tag{4.12}$$

since

$$\gamma_V^S(\sigma_x \mid +) = -\gamma_V^S(\sigma_x \mid -)$$

On the other hand, we have

$$\gamma_V^S(\sigma_x = 1 \mid +)/\big(1 - \gamma_V^S(\sigma_x = 1 \mid +)\big)$$
$$= \gamma_V^S(\sigma_x = 1 \mid +)/\gamma_V^S(\sigma_x = 1 \mid -) \leqslant e^{4\beta}$$

and therefore

$$\gamma_V^S(\sigma_x \mid +) \leqslant q = (e^{4\beta} - 1)/(e^{4\beta} + 1) < 1 \tag{4.13}$$

By iteration we obtain from (4.12) and (4.13)

$$\mu_S^+(\sigma_{(-N,0)}) \leqslant q^N$$

Thus we have shown that for all $N \geqslant 1$

$$P\Big(S : \inf_{x \in C_\infty^S} \mu_S^+(\sigma_x) \leqslant q^N\Big) > 0$$

This gives (3.15) because

$$S \to \inf_{x \in C_\infty^S} \mu_S^+(\sigma_x)$$

is invariant under translations and therefore P – almost surely constant. ∎

Now we turn to the uniqueness theorem 3.2.

Proof of Theorem 3.2. Because of (3.2) it is sufficient to prove

$$P(S : \mu_S^-(\sigma_x) = \mu_S^+(\sigma_x) \qquad \text{for all} \quad x \in S) = 1$$

Since

$$\mu_S^-(\sigma_x) \leqslant \mu_S^+(\sigma_x)$$

(this comes from the FKG Holley inequality, see Refs. 12 and 13), we only need to consider the quantities

$$\rho_+ = \int P(dS) 1_S(x) \mu_S^+(\sigma_x),\ \rho_- = \int P(dS) 1_S(x) \mu_S^-(\sigma_x)$$

(which are independent of x) and to show $\rho_+ = \rho_-$. This can be done following the lines of Lebowitz and Martin–Löf[13]: ρ_+ and ρ_- are considered as functions of the external field h; they are identified as the left and right derivatives of a convex function F of h which turns out to be differentiable at all $h \neq 0$. For the sake of completeness we sketch the argument. Let

$$F_\Lambda^\omega = \int P(dS)|\Lambda|^{-1} \log Z_{\Lambda\cap S}^S(\sigma_{S\setminus\Lambda}(\omega))$$

and

$$F = \lim_{\Lambda \uparrow L} F_\Lambda^\omega$$

where $\omega \in \Omega_L$ and the limit $\Lambda \uparrow L$ is taken over cubes. The existence of F and its independence of the choice of ω can be verified by the same methods as in the case when P is the unit mass on L; see Griffiths and Lebowitz[6] for the necessary modifications. Differentiation with respect to the external field gives

$$\frac{\partial}{\partial h} F_\Lambda^\omega = \int P(dS)|\Lambda|^{-1} \sum_{x \in \Lambda\cap S} \gamma_{\Lambda\cap S}^S(\sigma_x \,|\, \sigma_{S\setminus\Lambda}(\omega))$$

Both the FKG Holley inequality and Griffiths' second inequality show that

$$\gamma_{\Lambda\cap S}^S(\sigma_x \,|\, +)$$

is a decreasing function of Λ. Thus we obtain [letting $\Lambda(n)$ denote the cube of side length $(2n+1)$ which is centered at the origin]

$$\begin{aligned}
\rho_+ &\leqslant \lim_{n\to\infty} \frac{\partial}{\partial h} F_{\Lambda(n)}^+ \\
&= \lim_{n\to\infty} |\Lambda(n - n^{1/2})|^{-1} \\
&\quad \times \sum_{x \in \Lambda(n-n^{1/2})} \int P(dS) 1_{S-x}(0) \gamma_{[\Lambda(n)-x]\cap(S-x)}^{S-x}(\sigma_0 \,|\, +) \\
&\leqslant \lim_{n\to\infty} \int P(dS) 1_S(0) \gamma_{\Lambda(n^{1/2}/2-1)\cap S}^S(\sigma_0 \,|\, +) \\
&= \rho_+
\end{aligned}$$

Similarly,

$$\rho_- = \lim_{n\to\infty} \frac{\partial}{\partial h} F_{\Lambda(n)}^-$$

Moreover, all F_Λ^ω (and therefore also F) are convex functions of h; hence

$$\rho_- = \rho_+ = \frac{\partial}{\partial h} F \tag{4.14}$$

for all h at which F is differentiable. These are all $h \neq 0$. To see this one can use either the Lee–Yang circle theorem[6, 26] or, as observed by Preston,[27] the GHS inequality.[16] The latter asserts that $\gamma_{\Lambda\cap S}^S(\sigma_x \,|\, +)$ (and therefore also ρ_+) is a concave function of h in the region $h > 0$. Thus ρ_+ is continuous in this region which, combined with (4.14) and the convexity of F, implies (4.14) for all $h > 0$. Since $F(h) = F(-h)$ Theorem (3.2) follows. ■

Finally we give the proof of Theorem 3.1.

Proof of Theorem 3.1. Here we use a technique which is similar to that of Bricmont *et al.*[28] Without loss of generality we may assume that $S_2 = S_1 \cup \{x\}$ with $x \notin S_1$; moreover, we suppose $|S_1| = \infty$ because otherwise the theorem is trivial. For each $\Lambda_1 \subset S_1$ we write $\Lambda_2 = \Lambda_1 \cup \{x\}$.

(1) Let $\mu_1 \in \mathcal{G}_{S_1}(J)$. The backward Martingale convergence theorem implies that for μ_1 – almost all ω_1 and all finite $\Lambda_1 \subset S_1$, $\zeta \in \Omega_{\Lambda_1}$, and $a \in \{-1, 1\}$ the limits

$$\gamma_1(\sigma_{\Lambda_1} = \zeta \mid \omega_1) = \lim_{\Delta_1 \uparrow S_1} \gamma_{\Delta_1}^{S_1}(\sigma_{\Lambda_1} = \zeta \mid \sigma_{S_1 \setminus \Delta_1}(\omega_1)) \tag{4.15}$$

and

$$\begin{aligned} &\gamma_2(\sigma_x = a, \sigma_{\Lambda_1} = \zeta \mid \omega_1) \\ &\quad = \lim_{\Delta_1 \uparrow S_1} \gamma_{\Delta_2}^{S_2}(\sigma_x = a, \sigma_{\Lambda_1} = \zeta \mid \sigma_{S_2 \setminus \Delta_2}(\omega_1)) \\ &\quad = \lim_{\Delta_1 \uparrow S_1} \gamma_{\Delta_1}^{S_1}(1_{\{\sigma_{\Lambda_1} = \zeta\}} h_a \mid \sigma_{S_1 \setminus \Delta_1}(\omega_1)) / \gamma_{\Delta_1}^{S_1}(h_1 + h_{-1} \mid \sigma_{S_1 \setminus \Delta_1}(\omega_1)) \end{aligned} \tag{4.16}$$

exist; here

$$h_a(\omega) = \exp\Big(\sum_{X \ni x} J_X a \omega^{X \setminus \{x\}}\Big)$$

Furthermore,

$$\mu_1(\sigma_{\Lambda_1} = \zeta) = \int \mu_1(d\omega_1) \gamma_1(\sigma_{\Lambda_1} = \zeta \mid \omega_1)$$

We define $\mu_2 = \varphi(\mu_1)$ by the equation

$$\mu_2(\sigma_x = a, \sigma_{\Lambda_1} = \zeta) = \int \mu_1(d\omega_1) \gamma_2(\sigma_x = a, \sigma_{\Lambda_1} = \zeta \mid \omega_1)$$

It is easily verified that $\mu_2 \in \mathcal{G}_{S_2}(J)$.

From the inequality

$$\exp(-\|J\|_x) \leqslant h_a \leqslant \exp \|J\|_x$$

we obtain

$$\exp(-2\|J\|_x)\, \mu_1(\sigma_{\Lambda_1} = \zeta) \leqslant \mu_2(\sigma_{\Lambda_1} = \zeta) \leqslant \mu_1(\sigma_{\Lambda_1} = \zeta) \exp(2\|J\|_x)$$

for all Λ_1 and ζ; thus $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are equivalent on $\tilde{\mathcal{F}}_{S_1}$.

(2) Next we use the fact that $\mathcal{G}_{S_1}(J)$ is a simplex, i.e., each $\mu_1 \in \mathcal{G}_{S_1}(J)$ has a representation

$$\mu_1 = \int m(d\nu_1) \nu_1$$

by a probability measure m on the extreme elements of $\mathcal{G}_{S_1}(J)$ (see, for instance, Dynkin[29] or Theorem 2.1 of Preston[11]). It follows immediately from the definition of φ that

$$\varphi(\mu_1) = \int m(d\nu_1)\varphi(\nu_1)$$

Each extreme measure ν_1 in $\mathcal{G}_{S_1}(J)$ is trivial on the tail field on Ω_{S_1}; hence the equivalence of $\tilde{\nu}_1$ and $\varphi(\tilde{\nu}_1)$ implies $\tilde{\nu}_1 = \varphi(\tilde{\nu}_1)$ on

$$\tilde{\mathcal{F}}_\infty = \bigcap_{\Lambda \in S_0} \tilde{\mathcal{F}}_{S_1 \setminus \Lambda}$$

This gives $\tilde{\mu}_1 = \varphi(\tilde{\mu}_1)$ on $\tilde{\mathcal{F}}_\infty$.

(3) Suppose μ_1 and μ_1' are distinct elements of $\mathcal{G}_{S_1}(J)$. Then it is known[11] that even the restrictions of μ_1 and μ_1' to the tail field are distinct. Hence from the preceding we conclude $\varphi(\mu_1) \neq \varphi(\mu_1')$, i.e., φ is injective.

(4) A similar procedure as above gives us for each $\mu_2 \in \mathcal{G}_{S_2}(J)$ a measure $\mu_1 \in \mathcal{G}_{S_1}(J)$ such that $\tilde{\mu}_1 = \tilde{\mu}_2$ on $\tilde{\mathcal{F}}_\infty$. Then $\varphi(\mu_1) = \mu_2$ because $\varphi(\mu_1)$ and μ_2 belong to $\mathcal{G}_{S_2}(J)$ and coincide on tail events. Hence φ is surjective. ■

We conclude this section with the following proof.

Proof of (2.16). The "only if" part is obvious. If J is a ferromagnetic pair potential then the "if" part follows immediately from (3.1) and (3.2). In the general case we start from the following fact: If $S_n \to S$ (in the usual topology on $\mathbb{S}$ which comes from the identification $\mathbb{S} = \{0, 1\}^L$) then each weak limit point of any sequence $\tilde{\mu}_n$ with $\tilde{\mu}_n \in \tilde{\mathcal{G}}_{S_n}(J)$ belongs to $\tilde{\mathcal{G}}_S(J)$. Consequently, for each closed set F of probability measures on $\tilde{\Omega}$ the set

$$\{S \in \mathbb{S} : \tilde{\mathcal{G}}_S(J) \cap F \neq \emptyset\}$$

is closed and thus $\in \mathcal{Q}$. Therefore the conditions of a measurable choice theorem of Kuratowski and Ryll–Nardzewski (see the Corollary on p. 56 of Hildenbrand[30]) are satisfied; this states that there is a sequence of measurable mappings $S \to \tilde{\mu}_S^{(n)}$ from $\mathbb{S}$ (with $\mathcal{Q}$) to the set of all probability measures on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ (with the σ-algebra generated by the weakly open sets) such that, for each $S, \tilde{\mathcal{G}}_S(J)$ is the closure of $\{\tilde{\mu}_S^{(n)} : n \geq 1\}$. In particular,

$$\{S \in \mathbb{S} : |\tilde{\mathcal{G}}_S(J)| > 1\} = \{S \in \mathbb{S} : \tilde{\mu}_S^{(n)} \neq \tilde{\mu}_S^{(1)} \text{ for some } n\}$$

is measurable. Now assume that

$$P(S \in \mathbb{S} : |\mathcal{G}_S(J)| > 1) > 0$$

Then there is an n (say $n = 2$) and an $A \in \tilde{\mathcal{F}}$ such that $P(B) > 0$, where

$$B = \{S \in \mathbb{S} : \tilde{\mu}_S^{(1)}(A) > \tilde{\mu}_S^{(2)}(A)\}$$

Put

$$\tilde{\mu} = \int_B P(dS)\tilde{\mu}_S^{(1)} + \int_{S \setminus B} P(dS)\tilde{\mu}_S^{(2)}$$

and

$$\tilde{\mu}' = \int_B P(dS)\tilde{\mu}_S^{(2)} + \int_{\mathbb{S}\setminus B} P(dS)\tilde{\mu}_S^{(1)}$$

Then $\tilde{\mu}(A) > \tilde{\mu}'(A)$, and by construction $\tilde{\mu}$, $\tilde{\mu}' \in \tilde{\mathcal{G}}_P(J)$. ■

5. THE BOND MODEL

An inspection of the proofs in the previous section shows that, up to minor modifications, all results for the site model carry over to the bond model. We merely state the main results.

Theorem 5.1. Let J be any potential and $B_1, B_2 \in B$ with $|B_1 \Delta B_2| < \infty$. Then there is a bijection φ from $\mathcal{G}_{B_1}(J)$ to $\mathcal{G}_{B_2}(J)$ such that, for all $\mu_{B_1} \in \mathcal{G}_{B_1}(J)$, the measures μ_{B_1} and $\varphi(\mu_{B_1})$ are equivalent and coincide on

$$\bigcap_{\Lambda \in \mathbb{S}_0} \mathcal{F}^L_{L\setminus\Lambda}$$

Now suppose J is a ferromagnetic pair potential. Then each $\mathcal{G}_B(J)$ contains two particular measures μ_B^+ and μ_B^- which satisfy (3.1), (3.2), and (3.3) with S replaced by B. These are equal when J is translationally invariant with $h \neq 0$ and B is a typical realization of a stationary probability measure on $\mathfrak{B}$:

Theorem 5.2. Let J be a translationally invariant ferromagnetic pair potential with nonzero external field and P a translationally invariant probability measure on $\mathfrak{B}$. Then

$$|\mathcal{G}_B(J)| = 1$$

for P – almost all B.

Next we restrict ourselves to the two-dimensional case and consider the Ising potential J defined by (2.9) with $h = 0$. Here the results for the bond model are somewhat more complete than those for the site model because the bond percolation problem on the square lattice is better understood than the site problem. We say a set $C \in \mathbb{S}$ is B-connected if for all $x, y \in C$ there is a path $(x_1, \ldots, x_n)$ from x to y such that, for all $1 \leqslant k \leqslant n$, $\{x_k, x_{k+1}\} \in B$; such a path is called a B path. A maximal B-connected subset of L is called a B cluster. We let

$$\{\exists C_\infty^B\}$$

denote the set of all $B \in \mathfrak{B}$ for which an infinite B cluster exists. In dimension $d = 2$, Harris[31] has shown that

$$P_p(\exists C_\infty^B) = 0 \qquad \text{when} \quad p \leqslant \tfrac{1}{2} \tag{5.1}$$

and recently Kesten[32] has succeeded in proving the converse

$$P_p(\exists C_\infty^B) = 1 \qquad \text{when} \quad p > \tfrac{1}{2} \tag{5.2}$$

or, equivalently, $P_p(x \in C_\infty^B) > 0$ for all x when $p > \frac{1}{2}$; here $\{x \in C_\infty^B\}$ denotes the event that x belongs to an infinite B cluster. Moreover, it is known[31, 25] that for P_p – almost all B there is only one infinite B cluster; this will be denoted by C_∞^B.

Remark 5.1. Let $d = 2$ and J be given by (2.9) with $h = 0$ and $\beta > 0$. Then for all $p \leqslant \frac{1}{2}$ we have

$$\int P_p(dB)\mu_B^+(\sigma_x) = 0 \qquad \text{for all} \quad x$$

and

$$P_p(B : |\mathcal{G}_B(J)| = 1) = 1$$

However, for $p > \frac{1}{2}$ spontaneous magnetization occurs:

Theorem 5.3. Suppose $d = 2$ and J is the Ising potential with $h = 0$ and $\beta > 0$. Let $p > \frac{1}{2}$. Then

$$P_p\Big(B \in \mathfrak{B} : \lim_{\beta\to\infty} \mu_B^+(\sigma_x) = 1 \text{ for all } \quad x \in C_\infty^B\Big) = 1$$

In particular,

$$\lim_{\beta\to\infty} \int P_p(dB)\mu_B^+(\sigma_x) = P_p(x \in C_\infty^B) > 0 \qquad \text{for all} \quad x \in L$$

and there is a critical $\beta_c(p) < \infty$ such that

$$P_p(B \in \mathfrak{B} : |\mathcal{G}_B(J)| = 1) = 1 \qquad \text{when} \quad \beta < \beta_c(p)$$

and

$$P_p(B \in \mathfrak{B} : |\mathcal{G}_B(J)| > 1) = 1 \qquad \text{when} \quad \beta > \beta_c(p)$$

Moreover, if $\beta > \beta_c(p)$ then for P_p (almost all B) we have

$$\mu_B^+(\sigma_x) > 0 \qquad \text{for all} \quad x \in C_\infty^B$$

but

$$\inf_{x \in C_\infty^B} \mu_B^+(\sigma_x) = 0$$

Clearly, also in the present context the second Griffiths' inequality can be used to prove extensions of Theorem 5.3 which are similar to the Corollaries 3.1 and 3.2; their formulation is straightforward and therefore left to the reader. Also we do not dwell on the bond model counterpart of Remark 3.3. Instead we introduce some concepts which, in the proof of Theorem 5.3, replace those of Section 4.

First we define B regions for a given $B \in \mathfrak{B}$. A B path $(x_1, \ldots, x_n)$ in L is called a B circuit if $x_1 = x_n$. We say $\Lambda \in \mathcal{S}_0$ is a B region if Λ is the interior of a B circuit. If $p \geqslant \frac{1}{2}$ then for P_p – almost all B there is an increasing sequence $\Lambda_n(B)$ of B regions with $\bigcup_n \Lambda_n(B) = L$, see Lemma (3.6) of Smythe and Wierman[25].

Next we define B contours. B determines a unique subset B' of the set E' of edges in the dual lattice L' by the relations

$$B' = \{e'(e) : e \in B\}, B = \{e(e') : e' \in B'\}$$

We define B' paths, B' circuits, and B' clusters in L' just as we do B paths, B circuits, and B clusters in L.

Alternatively, a B' path is also said to be a B polygon, a B' circuit is called a $(B,0)$ contour, and $(L' \backslash B')$ clusters in L' are called $\overline{B}$ clusters. A (B,n) contour, $n \geqslant 1$, is an alternating sequence $g = (c_1, D_1, \ldots, c_n, D_n)$ with the following properties: $D_1, \ldots, D_n$ are pairwise distinct finite $\overline{B}$ clusters in L' and, for all $1 \leqslant k \leqslant n$, c_k is a B polygon from a site of D_{k-1} to a site of D_k (where $D_0 \equiv D_n$); see Fig. 2.

From these definitions it should be clear how the arguments in the proof of Theorem (3.3) can be fitted to the present context. Actually, the analog of estimate (4.7) is simpler than (4.7) because the concept of contiguity is not involved. The counterpart of Lemma 4.4 follows from the final remark in Section 3.6 of Smythe and Wierman[25].

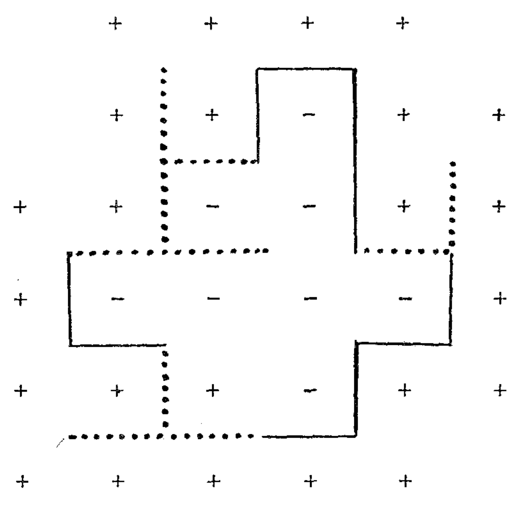

Fig. 2. A configuration ω of spins ($+$ or $-$) on the square lattice, and a set B of bonds; the edges in B are those which are not crossed by a dotted line. The dotted lines fall into three $\overline{B}$ clusters which, together with three B polygons (solid lines), form a $(B,3)$ contour g. g is realized by ω, and the B circuit at the boundary of the figure defines a B region containing g.

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